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NOTE

A Generalization of Fisher's Inequality

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In this paper we are concerned with the following conjecture. *Conjecture:* Let $\mathscr{L}$

$|\mathscr{A}| \leqslant \sum_{i=0}$ (i). In particular, we show this conjecture is true when $\mathscr{L}$ consists of k consecutive positive integers. This generalizes a well-known inequality of Fisher's. Our proof simplifies and extends a recent result of Ramanan's. © 1999 Academic Press

1. INTRODUCTION

Throughout this paper X will denote the set $[n]=\{1, 2, ..., n\}$ and $\mathscr{L}=\{l_1, l_2, ..., l_k\}$ will denote a set of k arbitrary nonnegative integers. Let $\mathscr{A}=\{A_1, A_2, ..., A_m\}$ denote a collection of subsets of X such that $|A_i \cap A_j| \in \mathscr{L}$. We will sometimes refer to $\mathscr{A}$ as an $\mathscr{L}$-*intersecting family*. In this paper we are, concerned with finding upper bounds for $|\mathscr{A}|$. Listed below (in chronological order) are, results related to this problem.

Theorem 1 [6]. *In a BIBD (balanced incomplete block design) the number of blocks is never less than the number of points.*

Theorem 2 [3]. *If every pair of sets in a uniform family has equal intersection size, then the number of sets does not exceed the number of points.*

Theorem 3 [4]. *If $\mathscr{L}=\{1\}$, then $|\mathscr{A}| \leqslant n$.*

Theorem 4 [11]. *If $\mathscr{L}=\{\lambda\}$ (for $1 \leqslant \lambda \leqslant n$), then $|\mathscr{A}| \leqslant n$.*

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THEOREM 5 [9]. *For any* $\mathscr{L}$, $|\mathscr{A}| \leqslant \sum_{i=0}^{k} \binom{n}{i}$.

THEOREM 6 [14]. *If* $\mathscr{L} = \{1, 2, ..., k\}$, *then* $|\mathscr{A}| \leqslant \sum_{i=0}^{k} \binom{n-1}{i}$.

It can be shown that Bose's result is equivalent to Fisher's result. Thus Theorems 1, 2, and 3 are special cases of Theorem 4 and Theorem 5 was a conjecture of Frankl's and Füredi's. In fact all of the above results are special cases of the following conjecture.

Conjecture 7 [16]. For any set $\mathscr{L}$ of k distinct positive integers $|\mathscr{A}| \leqslant \sum_{i=0}^{k} \binom{n-1}{i}$.

To see that Conjecture 7 implies Theorem 5, suppose that Conjecture 7 is true but Theorem 5 is false, i.e. suppose there exists a set $\mathscr{L}^0 = \{0 = l_1, l_2, ..., l_k\}$ and a set $\mathscr{A} = \{A_1, A_2, ..., A_m\}$ such that $|A_i \cap A_j| \in \mathscr{L}^0$, for all $A_i, A_j \in \mathscr{A}$ but $|\mathscr{A}| > \sum_{i=0}^{k} \binom{n}{i}$. Create a new collection of sets $\mathscr{A}' = \{A'_1, A'_2, ..., A'_m\}$ with $A'_i = A_i \cup \{n+1\}$. Then $\mathscr{A}'$ is a collection of subsets of $X' = X \cup \{n+1\}$ such that $|\mathscr{A}'_i \cap A'_j| \in \mathscr{L}^+ = \{1, l_2 + 1, ..., l_i + 1\}$, for all $A'_i, A'_j \in \mathscr{A}'$ and $|\mathscr{A}'| > \sum_{i=0}^{k} \binom{(n+1)-1}{i}$, which is a contradiction.

Although we have not proven Conjecture 7, we have managed to generalize Theorems 1 through 6, as Theorem 9 (given below) will show.

2. MAIN RESULT

Let $\mathscr{L} = \{l_1, l_2, ..., l_k\}$, where we shall assume throughout that $l_1 < l_2 < \cdots < l_k$. Let $g_{\mathscr{L}}(x) = \prod_{1 \leqslant i \leqslant k} (x - l_i)$, so that $g_{\mathscr{L}}(x) = 0$ if and only if $x \in \mathscr{L}$. Since $g_{\mathscr{L}}(x)$ is a monic polynomial in x of degree k, we can write it (by a change of basis) in the form $g_{\mathscr{L}}(x) = \sum_{h=0}^{k} c_h \binom{x}{h}$, where $c_0, c_1, ..., c_k$ are real numbers independent of x, which we call the *coefficients* of $\mathscr{L}$. Let $C^{\geqslant 0} = \{h\colon c_h \geqslant 0\}$ and $C^- = \{h\colon c_h < 0\}$. Clearly $k \in C^{\geqslant 0}$, since $c_k = k! > 0$.

Now suppose $\mathscr{A} = \{A_1, A_2, ..., A_m\}$ is an $\mathscr{L}$-intersecting family of subsets of $X = [n]$. With each $A_j \in \mathscr{A}$ we associate a real variable z_j. If $z = (z_1, z_2, ..., z_m)$ and $Y \subset X$ let $L_Y(z) = \sum_{j\colon Y \subset A_j} z_j$. Let $\binom{X}{h}$ be the set of all h-element subsets of X. We shall need the following lemma.

LEMMA 8. *If* $z = (z_1, z_2, ..., z_m)$ *and* $y = (y_1, y_2, ..., y_m)$, *then*

$$\sum_{h=0}^{k} c_h \sum_{Y \in \binom{X}{h}} L_Y(z)\, L_Y(y) = \sum_{j=1}^{m} g_{\mathscr{L}}(|A_j|)\, z_j y_j. \tag{1}$$

Proof. It is easy to see that

$$L_Y(z)\,L_Y(y)=\sum_{\substack{j\\ Y\subseteq A_j}} z_j y_j+\sum_{\substack{i\neq j\\ Y\subseteq A_i\cap A_j}} z_i y_j,$$

so that the LHS of (1) is equal to

$$\sum_{j=1}^{m}\sum_{h=0}^{k} c_h\binom{|A_j|}{h} z_j y_j+\sum_{\substack{i,\,j=1\\ i\neq j}}^{m}\sum_{h=0}^{k} c_j\binom{|A_i\cap A_j|}{h} z_i y_j$$

$$=\sum_{j=1}^{m} g_{\mathscr{L}}(|A_j|)\,z_j y_j+\sum_{\substack{i,\,j=1\\ i\neq j}}^{m} g_{\mathscr{L}}(|A_i\cap A_j|)\,z_i z_j,$$

which is equal to the RHS of (1) because $|A_i\cap A_j|\in\mathscr{L}$ and $g_{\mathscr{L}}(x)=0$ if $x\in\mathscr{L}$. ∎

We shall be concerned with the following two systems of linear equations:

$$L_Y(x)=0 \qquad \text{for all} \quad Y\subseteq X \qquad \text{with} \quad 0\leqslant|Y|\leqslant k \tag{S_1}$$

and

$$L_Y(z)=0 \qquad \text{for all} \quad Y\subset X \qquad \text{with} \quad |Y|\in C^{\geqslant 0}. \tag{S_2}$$

We are now ready to state and prove our main result.

THEOREM 9. *If $\mathscr{A}$ is an $\mathscr{L}$-intersecting family of subsets of X, then*

(a) *S_1 has only the trivial solution (hence $|\mathscr{A}|\leqslant\sum_{i=0}^{k}\binom{n}{i}$), and*

(b) *if $g_{\mathscr{L}}(|A_j|)\geqslant 0$ for all j, then $|\mathscr{A}|\leqslant\sum_{i\in C^{\geqslant 0}}\binom{n}{i}$.*

Proof. (a) Suppose that $\bar z=(\bar z_1\bar z_2, ..., \bar z_m)$ is solution to S_1. Set $z_j=\bar z_j$ for all j and $y_j=\bar z_j$ or $-\bar z_j$ according as $g_{\mathscr{L}}(|A_j|)\geqslant 0$ or <0. Then the LHS of (1) is 0, and the RHS is $\sum_{j=1}^{m}|g_{\mathscr{L}}(|A_j|)|\,\bar z_j^2$. We deduce immediately that $\bar z_j=0$ for all j with $g_{\mathscr{L}}(|A_j|)\neq 0$. In particular, this holds whenever $|A_j|>l_k$. Let $\mathscr{L}_i=\{l_1, ..., l_i\}$ for $i=1, ..., k$. Since $\bar z_j=0$ whenever $|A_j|>l_k$, the value of $L_Y(\bar z)$ is unchanged if we delete all such sets A_j from the family $\mathscr{A}$ thereby obtaining an $\mathscr{L}_{k-1}$ intersecting family. Repeating the above argument with this family, noting that $\bar z$ is evidently a solution to the subset of equations in S_1 corresponding to $0\leqslant|Y|\leqslant k-1$, we find now that $\bar z_j=0$ whenever $|A_j|>l_{k-1}$. Continuing in this way with $\mathscr{L}_{k-2}, ..., \mathscr{L}_1$, we find eventually that $\bar z_j=0$ for all j with $|A_j|>l_1$. But there can be at most one set in $\mathscr{A}$ of cardinality l_1, and since $L_\varnothing(\bar{z})=0$ we conclude that $\bar{z}_j=0$ for all j. This completes the proof of (a).

(b) Suppose now that $\bar{z}$ is a solution to S_2. Then setting $z_j=y_j=\bar{z}_j$ for each j, (1) becomes

$$\sum_{h\in C^-} c_y \sum_{Y\in\binom{x}{h}} L_Y(\bar{z})^2=\sum_{j=1}^{m} g_{\mathscr{L}}(|A_j|)\,\bar{z}_j^2. \tag{2}$$

Since the LHS of (2) is nonpositive and the RHS is nonnegative, both sides are 0, and so $\bar{z}$ is a solution to S_1. By (a) we deduce that S_2 has only the trivial solution, and (b) follows. ∎

We call $\mathscr{L}$ a *gte-set* (greater-than or-equal to set) if $g_{\mathscr{L}}(x)\geqslant 0$ for $x=l_1, l_1+1, ..., l_k$. We shall say that the coefficients of $\mathscr{L}$ *alternate in sign* if h is alternately in $C^{\geqslant 0}$ and in C^- for $h=k, k-1, ..., 0$.

COROLLARY 10. *Suppose that $\mathscr{A}$ is an $\mathscr{L}$-intersecting family and the coefficients of $\mathscr{L}$ alternate in sign. If $g_{\mathscr{L}}(|A_j|)\geqslant 0$ for all j (which must happen if $\mathscr{L}$ is a gte-set), then $|\mathscr{A}|\leqslant\sum_{i=0}^{k}\binom{n-1}{i}$.*

Proof. Clearly $|A_j|\geqslant l_1$ for all j, and $g_{\mathscr{L}}(|A_j|)>0$ if $|A_j|>l_k$, and so $g_{\mathscr{L}}(|A_j|)\geqslant 0$ for all j if $\mathscr{L}$ is a gte-set. Since $c_k\in C^{\geqslant 0}$ if the coefficients of $\mathscr{L}$ alternate in sign then (b) gives

$$|\mathscr{A}|\leqslant\sum_{i=0}^{\lfloor 1/2k\rfloor}\binom{n}{k-2i}=\sum_{i=0}^{n}\binom{n-1}{i}$$

by a well-known recurrence for binomial coefficients. ∎

COROLLARY 11. *Let $\mathscr{L}$ be a set of k consecutive positive integers, and suppose that $\mathscr{A}$ is an $\mathscr{L}$-intersecting family. Then $|\mathscr{A}|\leqslant\sum_{i=0}^{k}\binom{n-1}{i}$.*

Proof. Clearly $\mathscr{L}$ is a gte-set, and so it remains to show that the coefficients of $\mathscr{L}$ alternate in sign. Suppose $\mathscr{L}=\{m, m+1, ..., m+k-1\}$ where $m\geqslant 1$. Then $g_{\mathscr{L}}(x)=k!\binom{x-m}{k}$, and so the result follows from the identity

$$\binom{x-m}{k}=\sum_{i=0}^{k}(-1)^{k-i}\binom{m-1+k-i}{k-i}\binom{x}{i}. \tag{3}$$

To prove (3), choose x to be an integer greater than $m+k$ and equate the coefficients of y^k on the two sides of the equation

$$(1+y)^{x-m}=(1+y)^{-m}\,(1+y)^x.$$

Since (3) holds as an equation for all such x, it must be an identity. ∎

3. CONCLUSION

We should mention in [16] it was shown that, given any set $\mathscr{L}$ of k positive integers that Conjecture 7 is true when n is sufficiently large ($|X| = n$). We offer one more conjecture concerning the extremal families of Conjecture 7.

Conjecture 12. If $\mathscr{L}$ is a set of k consecutive positive integers and $\mathscr{A}$ is an $\mathscr{L}$ intersecting family with $|\mathscr{A}| = \sum_{i=0}^{k} \binom{n-1}{i}$, then $|\{|A|: A \in \mathscr{A}\}| \leqslant k+1$.

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